Metric Asymptotics on the Irregular Hitchin Moduli Space

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Outline

Higgs Bundles and the Hitchin Moduli Space

Higgs bundles

- C: compact Riemann surface.
- E: complex vector bundle over C.

Definition

A **Higgs bundle** is a pair $(\bar{\partial}_E, \varphi)$, where $\bar{\partial}_E$ is a holomorphic structure on E, and $\varphi \in \Omega^{1,0}(C, \mathrm{End}(E))$ with $\bar{\partial}_E \varphi = 0$.

Example

Suppose $\deg E = 0$, *and choose* $\bar{\partial}_E$ *so that* $\mathscr{E} := (E, \bar{\partial}_E) = K^{1/2} \oplus K^{-1/2}$. *Then there is a family of Higgs bundles given by*

$$\varphi_q = \begin{pmatrix} 0 & q \\ 1 & 0 \end{pmatrix},$$

parametrized by holomorphic quadratic differentials $q \in H^0(C, K^2)$.

Hitchin moduli space

Definition

A Higgs bundle $(\bar{\partial}_E, \varphi)$ is **(semi)stable**, if $\mu(\mathscr{F}) < \mu(\mathscr{E})$ (resp. $\leqslant$), for all φ-invariant holomorphic subbundle $\mathscr{F}$ of $\mathscr{E} = (E, \bar{\partial}_E)$.

Denote the rank and degree of E by r and d. The moduli space of stable Higgs bundles is $\mathscr{M}^{\mathrm{s}}(r,d) = \{(\bar{\partial}_E, \varphi) \text{ stable }\} / \mathscr{G}_{\mathbb{C}}$, where the complex gauge transformations $g \in \mathscr{G}_{\mathbb{C}}$ acts as $g \cdot (\bar{\partial}_E, \varphi) = (g^{-1} \circ \bar{\partial}_E \circ g, g^{-1}\varphi g)$.

- On $C = \mathbb{CP}^1$, $\mathscr{M}^{\mathrm{s}}(r,d) = \varnothing$.
- $\mathscr{M}^{\mathrm{s}}(1,d) = T^*\mathrm{Pic}^d(C)$.
- $\mathscr{M}^{\mathrm{ss}}(r,d)$ is a complex quasi-projective variety containing $\mathscr{M}^{\mathrm{s}}(r,d)$ as an open smooth subvariety of dimension $2 + 2r^2(g-1)$ (Nitsure, 1991).

Remark. When the structure group of E is G, one can define $G_{\mathbb{C}}$-Higgs bundles, then $\varphi \in \Omega^{1,0}(C, \mathfrak{g}_{\mathbb{C}}(E))$.

From now on we consider $\mathrm{SL}(r,\mathbb{C})$-Higgs bundles, then $\deg E = 0$, and the dimension of the moduli space becomes $2(r^2-1)(g-1)$.

Metrics

From Yang-Mills to Higgs

Self-dual Yang-Mills (Instanton)

$$\mathbb{R}^4 \qquad *F_A = F_A$$
$$A = \sum_{i=1}^4 A_i\, dx^i, F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j]$$
$$F_{12} = F_{34}, \quad F_{13} = -F_{24}, \quad F_{14} = F_{23}.$$

Assume that A is invariant under x^3- and x^4-translations. Let $z = x_1 + ix_2$, then $A = A_1\, dx^1 + A_2\, dx^2$ and $\Phi = \frac{1}{2}(A_3 - iA_4)dz$ satisfies (over $\mathbb{C}_z$)

$$\bar{\partial}_A \Phi = 0, \qquad F_A + [\Phi, \Phi^{*_{h_0}}] = 0. \tag{1}$$

The equations can be defined on any Riemann surface.

Theorem (Hitchin, 1987b; Simpson, 1988)

For $(\bar{\partial}_E, \varphi)$ stable, there exists a unique Hermitian metric h (called **harmonic metric**) solving the **Hitchin equation**

$$F_{D(\bar{\partial}_E, h)} + [\varphi, \varphi^{*_h}] = 0.$$

L^2 metric

Fix a Hermitian metric h_0 on E. Using the theorem we obtain (A, Φ) from stable $(\bar{\partial}_E, \varphi)$, where $g \cdot h = h_0, A = D(g^{-1}\bar{\partial}_E g, h_0), \Phi = g^{-1}\varphi g$. Then the Hitchin moduli space $\mathscr{M} = \{(\bar{\partial}_E, \varphi) \text{ stable }\} / \mathscr{G}_{\mathbb{C}}$ is isomorphic to

$$\{(A, \Phi) \text{ irreducible } | F_A + [\Phi, \Phi^{*_{h_0}}] = 0, \bar{\partial}_A \Phi = 0\} / \mathscr{G}.$$

Note that $(A, \Phi) \in \mathscr{C} = \mathscr{A}(E, h_0) \times \Omega^{1,0}(sl(E))$, the tangent space of $\mathscr{A}(E, h_0)$ is $\Omega^1(su(E)) \cong \Omega^{0,1}(sl(E))$. Define

$$g_{L^2}((\dot{A}_1^{0,1}, \dot{\Phi}_1), (\dot{A}_2^{0,1}, \dot{\Phi}_2)) = 2\mathrm{Re} \int_C \langle \dot{A}_1^{0,1}, \dot{A}_2^{0,1} \rangle_{h_0} + \langle \dot{\Phi}_1, \dot{\Phi}_2 \rangle_{h_0}.$$

$\mathscr{C}$ carries three constant complex structures

$$I(\alpha, \phi) = (\mathrm{i}\alpha, \mathrm{i}\phi), \;\; J(\alpha, \phi) = (\mathrm{i}\phi^{*_{h_0}}, -\mathrm{i}\alpha^{*_{h_0}}), \;\; K(\alpha, \phi) = (-\phi^{*_{h_0}}, \alpha^{*_{h_0}}).$$

Hyperkähler quotient

We let $\omega_\bullet = g_{L^2}(\bullet\cdot,\cdot)$, where $\bullet \in \{I, J, K\}$. Group of unitary gauge transformations $\mathscr{G}$ acts on $\mathscr{C}$ by isometries. This group action is **tri-Hamiltonian**: for each $\bullet$, there is a moment map $\mu_\bullet : \mathscr{C} \to (\mathrm{Lie}\,\mathscr{G})^*$ generating the action, i.e.,

$$d\mu_\bullet(\gamma)(\cdot) = \omega_\bullet(X_\gamma, \cdot)$$

where X_γ is the fundamental vector field generated by $\gamma \in \mathrm{Lie}\,\mathscr{G}$. Remarkably,

$$\mu_I(A, \Phi) = F_A + [\Phi, \Phi^{*_{h_0}}], \quad (\mu_J + \mathrm{i}\mu_K)(A, \Phi) = -2\mathrm{i}\bar{\partial}_A \Phi.$$

By the hyperkähler quotient construction of Hitchin, Karlhede, Lindström, and Roček (1987),

$$\mathscr{M} = \mu_I^{-1}(0) \cap \mu_J^{-1}(0) \cap \mu_K^{-1}(0)/\mathscr{G}$$

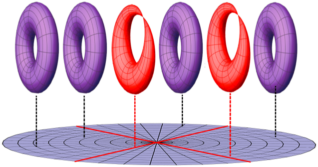
is a hyperkähler manifold. Moreover, g_{L^2} is **complete**.

Semiflat metric

Definition (Hitchin fibration, $\mathrm{SL}(2,\mathbb{C})$ case)

The **Hitchin fibration** (Hitchin, 1987a) is a surjective holomorphic map

$$\mathcal{H} : \mathcal{M} \to \mathcal{B} := H^0(K^2), [(\bar{\partial}_E, \varphi)] \mapsto \det \varphi.$$



$\mathcal{M}$ is completely integrable (generic fibers are Lagrangian tori), so by Freed (1999), it carries a **semiflat** metric g_{sf}, meaning that it is flat in the fiber direction and orthogonal to the base. g_{sf} is an **incomplete** hyperkähler metric.

Metric comparison

Theorem (Metric Comparison)

Fix $[(\bar{\partial}_E, \varphi)] \in \mathcal{M}'$, i.e., $\det \varphi$ has simple zeros. Then for infinitesimal deformations $[(\dot{\eta}, t\dot{\varphi})] \in T_{[(\bar{\partial}_E, t\varphi)]}\mathcal{M}'$ along the ray $[(\bar{\partial}_E, t\varphi)]$,

$$\|[(\dot{\eta}, t\dot{\varphi})]\|^2_{g_{L^2}} - \|[(\dot{\eta}, t\dot{\varphi})]\|^2_{g_{\mathrm{sf}}} = O(e^{-\epsilon t}), \text{ as } t \to \infty.$$

- Mazzeo, Swoboda, Weiss, and Witt (2019) proved polynomial decay for the $\mathrm{SL}(2, \mathbb{C})$ case.
- Dumas and Neitzke (2018) proved exponential decay along the Hitchin section in the $\mathrm{SL}(2, \mathbb{C})$ case.
- Fredrickson (2019) proved exponential decay in the $\mathrm{SL}(r, \mathbb{C})$ case.
- Fredrickson, Mazzeo, Swoboda, and Weiss (2020) proved exponential decay in the $\mathrm{SL}(2, \mathbb{C})$ parabolic case (Higgs field has simple poles).

Irregular Hitchin moduli spaces provide **more hyperkähler metrics**.

the Irregular Case

Irregular Hitchin moduli space

To define the irregular Hitchin moduli space, we need to fix the extra data:

- D: divisor on C.
- Singularity data near S, the support of D.
- parabolic structure at each $x \in S$. In $\mathrm{SL}(2,\mathbb{C})$ case, this means a choice of the filtration $\{0\} \subset L_x \subset E_x$ with weights $1 > \alpha_2 > \alpha_1 > 0$, $\alpha_1 + \alpha_2 = 1$.

As before, the moduli space $\mathscr{M}$ is $\{(\bar{\partial}_E, \varphi) \text{ stable}\}/\mathscr{G}_{\mathbb{C}}$, but now the gauge transformations should preserve the filtrations, and φ is meromorphic:

- $\varphi \in H^0(C, \mathrm{ParEnd}(\mathscr{E}) \otimes K(D))$.
- φ is compatible with the singularity data, meaning that in some local holomorphic trivialization of $\mathscr{E}$ near $x \in S$,

$$\varphi = \left(\frac{A_m}{z^m} + \cdots + \frac{A_1}{z} + \text{holomorphic terms} \right) \mathrm{d}z,$$

where A_j's are matrices fixed by the singularity data.

Metric comparison in the irregular case

Now we consider $\mathrm{SL}(2,\mathbb{C})$-Higgs bundles over $C = \mathbb{C}P^1$. In this case $\mathrm{tr}\varphi = 0$, $\mathrm{pdeg}\, E = \deg E + |S| = 0$, and $\dim_{\mathbb{C}} \mathcal{M} = 2(\deg D - 3)$. Previously defined hyperkähler metrics g_{L^2} and g_{sf} can be generalized to the irregular setting. Biquard and Boalch (2004) proved that g_{L^2} is a complete hyperkähler metric (for generic data).

Theorem (Chen and Li, 2022)

Fix a generic curve $[(\bar\partial_E, \varphi_t)]$ in $\mathcal{M}$, and an infinitesimal deformation $[(\dot\eta, \dot\varphi)] \in T_{[(\bar\partial_E,\varphi_t)]}\mathcal{M}$. As $t \to \infty$, there exist positive constants c, σ such that

$$\|[(\dot\eta, \dot\varphi)]\|^2_{g_{L^2}} - \|[(\dot\eta, \dot\varphi)]\|^2_{g_{\mathrm{sf}}} = O(\mathrm{e}^{-ct^\sigma}).$$

New features in the irregular case:

- In $\mathcal{M}$, there is no natural $\mathbb{C}^*$ action: $t \cdot [(\bar\partial_E, \varphi)] = [(\bar\partial_E, t\varphi)]$.
- Analysis near irregular singularities.

A curve in $\mathscr{M}$

For simplicity, we consider Higgs bundles with an (untwisted) order 4 pole. Recall the Hitchin fibration $\mathscr{H} : \mathscr{M} \to \mathscr{B}$, $[(\bar{\partial}_E, \varphi)] \to \det \varphi$. Now

$$\mathscr{B} = \left\{ \left(\sum_{k=5}^{8} \frac{\mu_k}{z^k} + \frac{t}{z^4} \right) dz^2 \middle| t \in \mathbb{C} \right\}.$$

Here μ_j's are constants determined by the singularity data. The pole is untwisted implies $\mu_8 \neq 0$, by rescaling, we assume $\mu_8 = -1$. Then we can find a curve $[(\bar{\partial}_E, \varphi_t)]$ parametrized by t, where $\mathscr{E} \cong \mathscr{O} \oplus \mathscr{O}(-1)$, and

$$\varphi_t = \frac{dz}{z^4} \begin{pmatrix} a_0(t) & b_t(z) \\ c_0 + z & -a_0(t) \end{pmatrix},$$

where $-a_0(t)^2 - b_t(z)(c_0 + z) = tz^4 + \cdots + \mu_7 z - 1 := \tilde{\nu}_t(z)$, and $-a_0(t)^2 = \tilde{\nu}_t(-c_0)$. Let $Z_t = \{z_j(t)\}_{k=1}^4$ be the zero set of $\tilde{\nu}_t(z)$, then $z_j(t) \sim t^{-1/4} e^{\mathrm{i}\pi(k-1)/2}$ (assume t is real and positive).

Idea of proof

We use a gluing process similar to that in Fredrickson et al. (2020). For $[(\dot{\eta}, \dot{\varphi})] \in T_{[(\bar{\partial}_E, \varphi_t)]}\mathcal{M}$, the L^2 metric can be expressed as

$$\|[(\dot{\eta}, \dot{\varphi})]\|_{g_{L^2}} = 2\int_C |\dot{\eta} - \bar{\partial}_E \dot{h}|^2_{h_t} + |\dot{\varphi} + [\dot{h}, \varphi]|^2_{h_t},$$

where $\dot{\eta}, \dot{\varphi}, \dot{h}$ satisfy the infinitesimal Hitchin equations and the Coulomb gauge condition. g_{sf} can be computed by the same formula with $h_t, \dot{h}$ replaced by $h_t^{\infty}, \dot{h}^{\infty}$. Here h_t^{∞} is the metric solving the decoupled Hitchin equations (pushfoward the HE-metric on the spectral line bundle)

$$F_{h_t^{\infty}} = 0, \quad [\varphi_t, \varphi_t^{*_{h_t^{\infty}}}] = 0.$$

Essentially, we only need to compare h_t and h_t^{∞}, in the following way

$$h_t^{\infty} \xrightarrow{\text{desingularize}} h_t^{\mathrm{app}} \xrightarrow{\text{perturb}} h_t.$$

Local forms (untwisted order four pole)

Locally in a disc of radius $\kappa t^{-1/4}$ around $x \in Z_t$, one can find holomorphic coordinate ζ such that $\det \varphi_t = -t^{9/4}\zeta\, \mathrm{d}\zeta^2$. Rescaling to get $-t^{3/2}\hat{\zeta}\mathrm{d}\hat{\zeta}^2$.

$$\bar{\partial}_E = \bar{\partial}, \quad \varphi_t = t^{3/4}\begin{pmatrix} 0 & 1 \\ \hat{\zeta} & 0 \end{pmatrix} \mathrm{d}\hat{\zeta}^2,$$

$$h_t^{\infty} = \begin{pmatrix} r^{1/2} & 0 \\ 0 & r^{-1/2} \end{pmatrix}, \quad h_t^{\mathrm{app}} = \begin{pmatrix} r^{1/2}\mathrm{e}^{\chi(r) l_{t^{3/4}}(r)} & 0 \\ 0 & r^{-1/2}\mathrm{e}^{-\chi(r) l_{t^{3/4}}(r)} \end{pmatrix},$$

where $r = |\hat{\zeta}|$, and $l_t(r) \lesssim e^{-ctr^{3/2}}$ solves the ODE

$$(\partial_r^2 + \partial_r / r) l_t = 8t^2 r \sinh(2l_t).$$

Near the order four pole, no desingularization is needed (h_t^{∞} is compatible with the parabolic structure): in a local holomorphic frame,
$\varphi_t = z^{-4}\sqrt{-\tilde{\nu}_t(z)}\mathrm{diag}(1,-1)\,\mathrm{d}z, \quad h_t^{\infty} = h_t^{\mathrm{app}} = \mathrm{diag}(|z|^{2\alpha_1}, |z|^{2\alpha_2})$.
Then $\|F_{h_t^{\mathrm{app}}} + [\varphi_t, \varphi_t^{*_{h_t^{\mathrm{app}}}}]\|_{L^2(h_t^{\mathrm{app}})} \leqslant c_1 e^{-c_2 t^{3/4}}$.

Modularity Conjecture

Gravitational instantons

Non-compact complete hyperkähler manifolds with $\int |\mathrm{Rm}|^2 < \infty$ are called gravitational instantons. As an improvement of earlier works of Minerbe (2010), Chen and Chen (2021b), Sun and Zhang (2023) have proved that any gravitational instanton must be ALE, ALF, ALG, ALH, ALG* or ALH*.

	Curvature	Volume	Tangent cone at infinity
ALE	$O(r^{-6})$	$O(r^4)$	$\mathbb{R}^4/\Gamma$
ALF-A_k	$O(r^{-3})$	$O(r^3)$	$\mathbb{R}^3$
ALF-D_k	$O(r^{-3})$	$O(r^3)$	$\mathbb{R}^3/\mathbb{Z}_2$
ALG	$O(r^{-2-\delta}), \delta = \min_{n\in\mathbb{Z}, n<2\beta} \frac{2\beta-n}{\beta}$	$O(r^2)$	$\mathbb{C}_\beta$
ALG*	$O(r^{-2}(\log r)^{-1})$	$O(r^2)$	$\mathbb{R}^2/\mathbb{Z}_2$
ALH	$O(e^{-\delta r})$	$O(r)$	$[0,\infty)$
ALH*	$O(r^{-2})$	$O(r^{4/3})$	$[0,\infty)$

Modularity Conjecture

Modularity Conjecture (Boalch)

Every Hitchin moduli space of dimension four is of type ALG or ALG*. Conversely, every ALG and ALG* gravitational instantons with Vol $\sim r^2$ can be realized as a Hitchin moduli space. The conjectural correspondence for ALG spaces is listed below. The tilde $(\tilde{\ })$ indicates a twisted irregular type.

	Regular	I_0^*	II	II^*
C	T_τ^2	$\mathbb{CP}^1$	$\mathbb{CP}^1$	$\mathbb{CP}^1$
D		$\{0, 1, p_0, \infty\}$	$\{0, 1, \infty\}$	$4 \cdot \{\tilde{0}\}$
G	$\mathrm{U}(1)$	$\mathrm{SU}(2)$	$\mathrm{SU}(6)$	$\mathrm{SU}(2)$
	III	III^*	IV	IV^*
C	$\mathbb{CP}^1$	$\mathbb{CP}^1$	$\mathbb{CP}^1$	$\mathbb{CP}^1$
D	$\{0, 1, \infty\}$	$4 \cdot \{0\}$	$\{0, 1, \infty\}$	$3 \cdot \{0\} + \{\infty\}$
G	$\mathrm{SU}(4)$	$\mathrm{SU}(2)$	$\mathrm{SU}(3)$	$\mathrm{SU}(2)$

Semiflat metric and model metric (untwisted order four pole)

Recall $\tilde{\nu}_t(z) = tz^4 + \cdots + \mu_7 z - 1 = t\prod_{k=1}^4 (z - z_k(t))$. The Hitchin fiber is (the compactification of) $\{-a_0^2 = \tilde{\nu}_t(-c_0) \mid (a_0, c_0) \in \mathbb{C}^2\}$, then it is isomorphic to $\mathbb{C}/(\mathbb{Z} \oplus \tau(t)\mathbb{Z})$. $\tau(t) = \lambda^{-1}(l(t))$, where $l(t)$ is the cross-ratio of $z_k(t)$'s. We have $\lim_{t\to\infty} \tau(t) = \mathrm{i}$.

The semiflat metric restricted to the Hitchin base (special Kähler metric) is

$$g_{\mathrm{sK}}(\dot{\nu}_t, \dot{\nu}_t) = \int_C \frac{|\dot{\nu}_t|^2}{|\nu_t|} d\mathrm{vol}_C = \frac{|\dot{t}|^2}{|t|} \int_{\mathbb{C}} \frac{1}{\left|\prod_{k=1}^4 (z - z_k(t))\right|} \mathrm{i}\, \mathrm{d}z \mathrm{d}\bar{z}$$

$$= (C_0 + O(r^{-1/2})) \frac{dr^2 + r^2 d\theta^2}{r^{1/2}},$$

where $r = |t|$. This is a conic metric with $\beta = 1 - (1/2)/2 = 3/4$.

ALG metrics from rank two irregular Hitchin moduli space

Kodaira type	II^*	III^*	IV^*
Dynkin diagram	A_0	A_1	A_2
β	$\frac{5}{6}$	$\frac{3}{4}$	$\frac{2}{3}$
τ	$e^{2\pi \mathrm{i}/3}$	i	$e^{2\pi \mathrm{i}/3}$
D	$4 \cdot \{\tilde{0}\}$	$4 \cdot \{0\}$ or $3 \cdot \{\tilde{0}\} + \{\infty\}$	$3 \cdot \{0\} + \{\infty\}$

Table. ALG

β, τ means that a dense set of $\mathscr{M}$ is asymptotic to

$$\{z \in \mathbb{C}, \arg z \in (0, 2\pi\beta)\} \times \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau).$$

The Kodaira type means that $\mathscr{M}$ is biholomorphic to (Chen and Chen, 2021a) a rational elliptic surface minus a fiber with the given Kodaira type. Dynkin diagram means that $H^2(\mathscr{M})$ is generated by the given extended Dynkin diagram. This makes sense because any ALG gravitational instanton with the same β is diffeomorphic to each other (Chen and Viaclovsky, 2021).

ALG* metrics

Kodaira type	I_4^*	I_3^*	I_2^*	I_1^*
Dynkin diagram	D_0	D_1	D_2	$D_3 = A_3$
D	$2\cdot\{\tilde{0}\}$ $+2\cdot\{\tilde{\infty}\}$	$2\cdot\{0\}$ $+2\cdot\{\tilde{\infty}\}$	$2\cdot\{0\}+2\cdot\{\infty\}$ or $\{0\}+\{1\}+2\cdot\{\tilde{\infty}\}$	$\{0\}+\{1\}$ $+2\cdot\{\infty\}$

Table. ALG*

The Kodaira type means that $\mathscr{M}$ is biholomorphic to (Chen and Viaclovsky, 2021) a rational elliptic surface minus a fiber with the given Kodaira type. Dynkin diagram means that $H^2(\mathscr{M})$ is generated by the given extended Dynkin diagram. This makes sense because any ALG* gravitational instanton with the same Kodaira type at infinity is diffeomorphic to each other (Chen and Viaclovsky, 2021).

Thank you for your attention!

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